The Black Hole in Three Dimensional Spacetime

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Abstract

The standard Einstein-Maxwell equations in 2+1 spacetime dimensions, with a negative cosmological constant, admit a black hole solution. The 2+1 black hole -characterized by mass, angular momentum and charge, defined by flux integrals at infinity- is quite similar to its 3+1 counterpart. Anti-de Sitter space appears as a negative energy state separated by a mass gap from the continuous black hole spectrum. Evaluation of the partition function yields that the entropy is equal to twice the perimeter length of the horizon.

PACS number: 04.20.Cv, 04.20.Fy, 04.20.Jb, 04.60.+n.

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The fascinating properties of the black hole, classical and -especially-quantum, have made it long desirable to have available a lower dimensional analog which could exhibit the key features without the unnecessary complications.

It is the purpose of this letter to report that the sought for analog does exist in standard 2+1 Einstein-Maxwell theory with a negative cosmological constant.

For simplicity we will first ignore the coupling to the Maxwell field. The generalization to non-zero electric charge will be indicated afterwards.

The action is

$$I = \frac{1}{2\pi} \int \sqrt{-g} \left[R + 2l^{-2} \right] d^2x dt + B,$$ (1)

where B is a surface term and the radius l is related to the cosmological constant by $-\Lambda = l^{-2}$.

The equations of motion derived from (1) are solved by the black hole field

$$ds^2 = -N^2 dt^2 + N^{-2} dr^2 + r^2 (N^\phi dt + d\phi)^2$$ (2)

where the squared lapse $N^2(r)$ and the angular shift $N^\phi(r)$ are given by

$$N^2(r) = -M + \frac{r^2}{l^2} + \frac{J^2}{4r^2}$$

$$N^\phi(r) = -\frac{J}{2r^2}$$

with $-\infty < t < \infty$, $0 < r < \infty$ and $0 \le \phi \le 2\pi$.

In this letter we will focus our attention mainly on the physical properties of the solution. The geometric structure will be briefly touched upon at the end and its detailed study will be provided in a forthcoming publication[1].

The two constants of integration M and J appearing in (2) are the conserved charges associated with asymptotic invariance under time displacements (mass) and rotational invariance (angular momentum), respectively. These charges are given by flux integrals through a large circle at spacelike infinity.

The lapse function $N(r)$ vanishes for two values of r given by

$$r_\pm = l \left[\frac{M}{2} \left(1 \pm \sqrt{1 - \left(\frac{J}{Ml} \right)^2} \right) \right]^{1/2}.$$

Of these, r_+ is the black hole horizon. In order for the horizon to exist one must have

$$M > 0, \quad |J| \le Ml. \tag{3}$$

In the extreme case $|J| = Ml$, both roots of $N^2 = 0$ coincide.

Note that the radius of curvature $l = (-\Lambda)^{-1/2}$ provides the length scale necessary in order to have a horizon in a theory in which the mass is dimensionless. If one lets l grow very large the black hole exterior is pushed away to infinity and one is left just with the inside.

The vacuum state, namely what is to be regarded as empty space, is obtained by making the black hole disappear. That is, by letting the horizon size go to zero. This amounts to letting $M \to 0$, which requires $J \to 0$ on account of (3). One thus obtains the line element

$$ds^2_{vac} = -(r/l)^2 dt^2 + (r/l)^{-2} dr^2 + r^2 d\phi^2. \tag{4}$$

As M grows negative one encounters the solutions studied previously in Refs. [2, 3]. The conical singularity that they posses is naked, just as the curvature singularity of a negative mass black hole in $3{+}1$ dimensions. Thus, they must, in the present context, be excluded from the physical spectrum. There is however an important exceptional case. When one reaches $M = -1$ and $J = 0$ the singularity dissapears. There is no horizon, but there is no singularity to hide either. The configuration

$$ds^2 = -(1 + (r/l)^2) dt^2 + (1 + (r/l)^2)^{-1} dr^2 + r^2 d\phi^2$$

(anti-de Sitter space) is again permissible.

Therefore, one sees that anti-de Sitter space emerges as a "bound state", separated from the continuous black hole spectrum by a mass gap of one unit. This state cannot be deformed continuously into the vacuum (4), because the deformation would require going through a sequence of naked singularities which are not included in the configuration space.

Note that the zero point of energy has been set so that the mass vanishes when the horizon size goes to zero. This is quite natural. It is what is done in 3+1 dimensions. In the past, the zero of energy has been adjusted so that, instead, anti-de Sitter space has zero mass. Quite appart from this diference, the key point is that the black hole spectrum lies above the limiting case $M = 0$.

The 2+1 black hole has thermodynamic properties similar to those found in $3 + 1$ dimensions[4]. In the steepest descent approximation, the free energy F divided by the temperature is given by the value of the Euclidean action evaluated on the Euclidean continuation[5] of the black hole field (2). The surface terms appearing in the action are here crucial. They must be constructed so that the action truly has an extremum on the class of fields considered[6]. In the variation one must allow changes in the fields contributing to the surface integrals giving M and J, but must hold fixed their momenta (appropiate variational derivatives of the action on the boundaries),

which become the "thermodynamical conjugates"[7]. These conjugates are the period β of the Euclidean Killing time (inverse temperature T^{-1}) and the rotational chemical potential -which turns out to be the negative of the angular shift N^ϕ evaluated on the horizon ("angular velocity").

To determine the surface terms, we found it best, both for conceptual and practical reasons, not to work with the covariant form of the action (1) but to start instead with its Hamiltonian version

$$I' = \int \left[\pi^{ij} \dot{g}_{ij} - N\mathcal{H} - N^i \mathcal{H}_i \right] d^2x dt + B'.$$

The surface term B' differs from B in (1) (the volume integrals of I and I' differ by a surface term).

Working with the Hamiltonian action has the following advantages: (i) Since the metric is time independent, the value of the volume piece of the Hamiltonian action is equal to zero when the constraints hold. Thus, the surface terms are everything, even in the presence of the cosmological constant. (ii) One knows right away the surface term that must be added at infinity without need to regularize. For the Euclidean action, it is simply the period β of Killing time multiplied by the mass (by definition of the mass).

After infinity has been dealt with, there remains only to make sure that the variational derivative of the action should vanish on the horizon. This

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makes it necessary to include in B'_{Euc} two further "surface terms" at $r = r_+$. They turn out to be equal to minus two times the proper perimeter length of the horizon (to cancel the variation of the hamiltonian constraint) and $\beta N^\phi(r_+)J$ (to cancel the variation of the momentum constraint).

One thus gets for the Euclidean action

$$I_{Euc} = \beta M - 4\pi r_+ + \beta N^\phi(r_+)J. \tag{5}$$

But, $I_{Euc} = F/T$, where the free energy is $F = M - TS - \sum \mu_i C_i$ and the μ's are the chemical potentials thermodynamically conjugate to the conserved charges C_i. Therefore, equation (5) confirms that β and $-N^\phi(r_+)$ are the inverse temperature and the chemical potential corresponding to J, respectively. It also shows that the entropy is equal to twice the perimeter length of the horizon,

$$S = 2L = 4\pi r_+. \tag{6}$$

From (6), one may evaluate the temperature of the black hole,

$$T = \left[\frac{\partial S}{\partial M}\right]_J^{-1} = \frac{r_+^2 - r_-^2}{2\pi r_+}.$$

This expression coincides with the periodicity in Euclidean Killing time needed to make the Euclidean black hole geometry regular at the horizon.

One may also verify that $N^\phi(r_+) = T(\partial S/\partial J)_M$.

Note that as the horizon disappears, the temperature goes to zero in contrast with the $3+1$ case. On the other hand, the extreme rotating hole $(J = Ml)$ has zero temperature and non-zero entropy, just as the 3+1 case.

Now, we briefly discuss how the electromagnetic field is brought in. One includes the following additional contributions in the action: (i) The electromagnetic energy and momentum densities are added to $\mathcal{H}$ and $\mathcal{H}_i$ respectively, (ii) A term $\pi^i \dot{A}_i$ is added to $\pi^{ij}\dot{g}_{ij}$, (iii) The Gauss law constraint is incorporated by adding $+\int d^2x dt A_0 \pi^i_{,i}$ to the volume piece of the action. (iv) This makes it mandatory to include in B'_{Euc} a new surface integral equal to $A_0(r_+)Q$. Here Q is the electric charge given by a flux integral at infinity, and equal to the constant value throughout space of the radial component π^r.

The only non vanishing component of the electromagnetic vector potential may be taken to be

$$A_0(r) = -Q ln(r/r_0).$$

The only modification of the metric (2) is that the lapse function in (3) must be replaced by

$$N^2 = N^2_{(Q=0)} + \tfrac{1}{2}QA_0(r).$$

The free energy acquires an extra term $-A_0(r_+)Q$ and the entropy is again equal to twice the proper perimeter length of the black hole. The horizon exists for any value of Q provided the bound (3) on J is obeyed[8].

Lastly, we turn to some comments on the geometry of the black hole. For simplicity, these comments are restricted to $Q = 0$ (no Maxwell field). In that case, one is dealing with a spacetime of constant negative curvature (the Riemann tensor is a constant multiple of an antisymmetrized product of metric tensors). It is well known [9] that such a space time must arise from identifications of points in anti de Sitter space through a discrete subgroup of its symmetry group $O(2,2)$. In this case, the discrete subgroup is generated by one element, the exponential of a particular Killing vector. In terms of the embedding

$$-u^2 - v^2 + x^2 + y^2 = -l^2$$

of anti de Sitter space in flat four dimensional space that Killing vector is given by

$$\xi = \frac{r_+}{l}\left(x\frac{\partial}{\partial u} + u\frac{\partial}{\partial x}\right) - \frac{r_-}{l}\left(y\frac{\partial}{\partial v} + v\frac{\partial}{\partial y}\right).$$

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Throughout anti de Sitter space this vector can be spacelike, null or timelike. The whole of the black hole geometry is the region where ξ is spacelike. This region is incomplete. Its boundaries are the surfaces $\xi^2 = 0$ which correspond to $r = 0$ in the metric (2). One cannot continue past these boundaries because ξ becomes timelike and the identification would produce closed timelike lines.

The rich structure of the $2 + 1$ black hole is remarkable given the simple nature of gravitation in three spacetime dimensions[10]. One may hope that its study will provide further understanding of the black hole, especially at the quantum level.

Note Added: The charged, rotating solution as described in this Letter is incorrect, as pointed out by several authors [11]. The correct metric can be found in [12] and had been independently obtained in a different approach in [13].

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[5] The Euclidean continuation may be obtained by setting $t = -i\tau$ in (2). This makes the metric complex when $J \neq 0$. The Euclidean action is given by $iI = -I_{Euc}$. Although correct in the case at hand, making imaginary a particular coordinate in a generally covariant theory

is disturbing. A more satisfactory way to define the Euclidean continuation is the following. One solves the Euclidean field equations but introduces complex constants of integration in the solutions, thus generating complex solutions to real equations. In the present example, one sets $J_{Euc} = -iJ$, and, when there is electric charge, $Q_{Euc} = -iQ$. Here J and Q are the physical (Minkowskian signature) values of the angular momentum and electric charge. This point of view makes it clear that the procedure is coordinate invariant.

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scured the fact that the entropy itself is a surface term at the horizon. Expressing the entropy as a surface integral on the horizon plays a key role in a ongoing program by F. Wilczek [Princeton lectures notes, unpublished] to elucidate the physical origin of the black hole entropy. We thank him for inspiring discussions on black hole physics.

[8] The constant r_0 is an arbitrarly reference point for the energy, which cannot be taken to be at infinity as in the $3+1$ case. If r_0 is replaced by r_0', M is replaced by $M + \frac{1}{2}Q^2 ln(r_0'/r_0)$. However, the variation of the free energy keeping the temperature and the chemical potentials fixed is independent of r_0, and so are, T, S and $N^\phi(r_+)$.

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